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# On Alexander polynomials of some reduced curves

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## 1 Introduction

Let  $C$  be a plane curve in  $\mathbb{P}^2$ . We are interested in several topological invariants. In this report, we consider *Alexander polynomials of some reduced curves* which is defined as the following. Let  $Q$  be a reduced quartic. Suppose that  $Q$  has at most  $A_1$  singularities. Let  $C_1, \dots, C_n$  be smooth conics such that:

1. Each  $C_i$  is tangent to  $Q$  with intersection multiplicity 2 at 4 smooth points for any  $i$ .
2. For all pairs  $(i, j)$  ( $i \neq j$ ),  $C_i$  intersects transversely with  $C_j$  at all intersection points.
3.  $C_i \cap C_j \cap C_k = \emptyset$  and  $C_i \cap C_j \cap Q = \emptyset$ .

Let  $B := Q + C_1 + \dots + C_n$  be the reduced curve which consists of the above quartic and smooth conics and let  $Q \cap C_i = \{P_{i1}, \dots, P_{i4}\}$  be the tangent points of  $C_i$  and  $Q$  for  $i = 1, \dots, n$ . Note that the configurations of singularities of  $B$  is

$$\Sigma(B) = \Sigma(Q) + \{2n(n-1)A_1, 4nA_3\}.$$

For example, in [7], Namba and Tsuchihashi considered the case  $n = 2$  and  $Q$  is a union of two smooth conics which are intersects transversely each other.

In this report, we consider the case  $n = 3$  and then we determine their Alexander polynomials.

## 2 Alexander polynomials

### 2.1 Definition of Alexander Polynomials

Let  $C$  be an affine curve of degree  $d$ . Suppose that the line at infinity  $L_\infty$  intersects transversely with  $C$ . Let  $\phi : \pi_1(X) \rightarrow \mathbb{Z}$  be the composition of Hurewicz homomorphism and the summation homomorphism. Let  $t$  be a generator of  $\mathbb{Z}$  and we put the Laurent polynomial ring  $\Lambda := \mathbb{C}[t, t^{-1}]$ . We consider an infinite cyclic covering  $p : \tilde{X} \rightarrow X$  such that  $p_*(\pi_1(\tilde{X})) = \ker \phi$ . Then  $H_1(\tilde{X}, \mathbb{C})$  has a structure of  $\Lambda$ -module. Thus we have

$$H_1(\tilde{X}, \mathbb{C}) = \Lambda/\lambda_1(t) \oplus \cdots \oplus \Lambda/\lambda_m(t)$$

where we can take  $\lambda_i(t) \in \Lambda$  is a polynomial in  $t$  such that  $\lambda_i(0) \neq 0$  for  $i = 1, \dots, m$ . The Alexander polynomial  $\Delta_C(t)$  is defined by the product  $\prod_{i=1}^m \lambda_i(t)$ .

In this report, we use the *Loeser-Vaquié formula* ([10, 9]) for calculating Alexander polynomials. Hereafter we follow the notations and terminologies of [4, 9] for the Loeser-Vaquié formula.

### 2.2 Loeser-Vaquié formula

Let  $[X, Y, Z]$  be homogenous coordinates of  $\mathbb{P}^2$  and let us consider the affine space  $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z = 0\}$  with affine coordinates  $(x, y) = (X/Z, Y/Z)$ . Let  $f(x, y)$  be the defining polynomial of  $C$ . Let  $\text{Sing}(C)$  be the singular locus of  $C$  and let  $P \in \text{Sing}(C)$  be a singular point. Consider a resolution  $\pi : \tilde{U} \rightarrow U$  of  $(C, P)$ , and let  $E_1, \dots, E_s$  be the exceptional divisors of  $\pi$ . Let  $(u, v)$  be a local coordinate system centered at  $P$  and  $k_i$  and  $m_i$  be respective order of zero of the canonical two form

$\pi^*(du \wedge dv)$  and  $\pi^*f$  along the divisor  $E_i$ . The adjunction ideal  $\mathcal{J}_{P,k,d}$  of  $\mathcal{O}_P$  is defined by

$$\mathcal{J}_{P,k,d} = \left\{ \phi \in \mathcal{O}_P \mid (\pi^*\phi) \geq \sum_i (\lfloor km_i/d \rfloor - k_i) E_i \right\}, \quad k = 1, \dots, d-1$$

where  $\lfloor * \rfloor = \max\{n \in \mathbb{Z} \mid n \leq *\}$  and we call it the floor function.

Let  $O(j)$  be the set of polynomials in  $x, y$  whose degree is less than or equal to  $j$ . We consider the canonical mapping  $\sigma : \mathbb{C}[x, y] \rightarrow \bigoplus_{P \in \text{Sing}(C)} \mathcal{O}_P$  and its restriction:

$$\sigma_k : O(k-3) \rightarrow \bigoplus_{P \in \text{Sing}(C)} \mathcal{O}_P.$$

Put  $V_k(P) := \mathcal{O}_P / \mathcal{J}_{P,k,d}$  and denote the composition of  $\sigma_k$  and the natural surjection  $\bigoplus \mathcal{O}_P \rightarrow \bigoplus V_k(P)$  by  $\bar{\sigma}_k$ . Then the Alexander polynomial of  $C$  is given as follows:

**Theorem 1.** ([5, 6, 1, 3]) *The reduced Alexander polynomial  $\tilde{\Delta}_C(t)$  is given by the product*

$$\tilde{\Delta}_C(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k}, \quad \ell_k := \dim \text{coker } \bar{\sigma}_k \quad (1)$$

where

$$\Delta_k(t) = \left( t - \exp\left(\frac{2k\pi i}{d}\right) \right) \left( t - \exp\left(-\frac{2k\pi i}{d}\right) \right).$$

The Alexander polynomial  $\Delta_C(t)$  is given as

$$\Delta_C(t) = (t-1)^{r-1} \tilde{\Delta}_C(t)$$

where  $r$  is the number of irreducible components of  $C$ .

## 2.3 The adjunction ideal for non-degenerate singularities

In general, the computation of the ideal  $\mathcal{J}_{P,k,d}$  requires an explicit computation of the resolution of the singularity  $(C, P)$ . However for the case

of non-degenerate singularities, the ideal  $\mathcal{J}_{P,k,d}$  can be obtained combinatorially by a toric modification. Let  $(u, v)$  be a local coordinate system centered at  $P$  such that  $(C, P)$  is defined by a function germ  $f(u, v)$  and the Newton boundary  $\Gamma(f; u, v)$  is non-degenerate. Let  $R_1, \dots, R_s$  be the primitive weight vectors which correspond to the faces  $\Delta_1, \dots, \Delta_s$  of  $\Gamma(f; u, v)$ . Let  $\pi : \tilde{U} \rightarrow U$  be the canonical toric modification and let  $\hat{E}(R_i)$  be the exceptional divisor corresponding to  $R_i$ . Recall that the order of zeros of the canonical two form  $\pi^*(du \wedge dv)$  along the divisor  $\hat{E}(R_i)$  is simply given by  $|R_i| - 1$  where  $|Q_i| = p + q$  for a weight vector  $R_i = {}^t(p_i, q_i)$  (see [8]). For a function germ  $g(u, v)$ , let  $m(g, R_i)$  be the multiplicity of the pull-back  $(\pi^*g)$  on  $\hat{E}(R_i)$ . Then

**Lemma 1** ([2, 9]). *A function germ  $g \in \mathcal{O}_P$  is contained in the ideal  $\mathcal{J}_{P,k,d}$  if and only if  $g$  satisfies following condition:*

$$m(g, R_i) \geq \left\lfloor \frac{k}{d} m(f, R_i) \right\rfloor - |R_i| + 1, \quad i = 1, \dots, s.$$

*The ideal  $\mathcal{J}_{P,k,d}$  is generated by the monomials satisfying the above conditions.*

**Example 1.** *Let  $C$  be a plane curve of degree  $2n + 4$  such that  $C$  has only  $A_1$  and  $A_3$  singularities. Let  $P$  be a singular point.*

(1) *Assume that  $P$  is an  $A_1$  singularity. Then the adjunction ideal is*

$$\mathcal{J}_{P,k,2n+4} = \left\langle u^a v^b \mid a + b \geq \left\lfloor \frac{k}{n+2} \right\rfloor - 1 \right\rangle = \mathcal{O}_P,$$

*for all  $k = 3, \dots, 2n + 3$ . Hence  $A_1$  singularity do not contribute to computations of Alexander polynomials because  $V_P(k)$  is 0 for all  $k$ .*

(2) *Assume that  $P$  is an  $A_3$  singularity. Then the adjunction ideal is*

$$\begin{aligned} \mathcal{J}_{P,k,2n+4} &= \left\langle u^a v^b \mid a + 2b \geq \left\lfloor \frac{2k}{n+2} \right\rfloor - 2 \right\rangle \\ &= \begin{cases} \mathcal{O}_P, & 3 \leq k < \left\lceil \frac{3}{2}(n+2) \right\rceil, \\ m_P, & \left\lceil \frac{3}{2}(n+2) \right\rceil \leq k \leq 2n+3 \end{cases} \end{aligned}$$

*where  $\lceil * \rceil = \min\{n \in \mathbb{Z} \mid * \leq n\}$  and we call it the ceil function.*

## 2.4 The Alexander polynomials of subconfigurations of reduced curves

Let  $B$  be a reduced curve on  $\mathbb{P}^2$ ,  $B = B_1 + \cdots + B_r$  be its irreducible decomposition. Let  $I$  be a non-empty subset of the index  $J := \{1, \dots, r\}$  and  $B_I := \sum_{i \in I} B_i$ . We define  $\mathbf{Alex}(B)$  as follows:

$$\mathbf{Alex}(B) = (\Delta_{B_I}(t))_{I \in 2^J \setminus \{\emptyset\}}.$$

Clearly,  $\mathbf{Alex}(B)$  is a topological invariant of  $(\mathbb{P}^2, B)$ . We also define  $\widetilde{\mathbf{Alex}}(B)$  as the set of the reduced Alexander polynomials of subconfigurations of  $B$ . We consider subsets of  $\widetilde{\mathbf{Alex}}(B)$ :

$$\widetilde{\mathbf{Alex}}(B)_s := (\tilde{\Delta}_{B_I}(t))_{\#I=s}, \quad 1 \leq s \leq r.$$

We say that  $\widetilde{\mathbf{Alex}}(B)_s$  is *trivial* if any reduced Alexander polynomial in  $\widetilde{\mathbf{Alex}}(B)_s$  is 1.

For our curves,  $\widetilde{\mathbf{Alex}}(B)_1$  and  $\widetilde{\mathbf{Alex}}(B)_2$  are trivial.

Now we consider  $\widetilde{\mathbf{Alex}}(B)_s$  where  $s \geq 3$ . Let  $I$  be a non-empty subset of  $\{1, \dots, n+1\}$ . We correspond  $n+1$  to the quartic  $Q$ . If  $n+1 \notin I$ , then  $\tilde{\Delta}_{B_I}(t) = 1$  as  $B_I$  has only  $A_1$  singularities. Hence we consider the Alexander polynomial of  $B_I$  where  $I$  contains  $n+1$ .

To determine the Alexander polynomial of  $B_I$ , we consider the adjunction ideals and the map  $\bar{\sigma}_k : O(k-3) \rightarrow V(k)$ . The adjunction ideals for each singular point are computed in Example 1. Now we consider the multiplicity  $\ell_k$  in the formula (1) of Loeser-Vaqu   which is given as

$$\ell_k = \dim \operatorname{coker} \bar{\sigma}_k = \rho(k) + \dim \ker \bar{\sigma}_k.$$

where  $\rho(k) = \sum_{P \in \operatorname{Sing}(B_I)} \dim V_k(P) - \dim O(k-3)$ . For fixed  $k$ , the integer  $\rho(k)$  is determined by only the adjunction ideal. Hence we should consider the dimension of  $\ker \bar{\sigma}_k$ .

By Lemma 1, if  $k < \lceil \frac{3}{2}(n+2) \rceil$ , then  $V(k) = 0$ . That is  $\ell_k = 0$ . For other cases,  $V(k) = \mathbb{C}^{4n}$  and  $g \in \ker \bar{\sigma}_k$  if and only if  $\{g = 0\}$  passes through all  $A_3$  singular point  $P$ . Hence we investigate the linear series  $\mathcal{N}_{k-3}(\mathcal{P})$ . In general, the dimension of  $\mathcal{N}_{k-3}(\mathcal{P})$  is greater than or equal to  $N := \frac{(k-2)(k-1)}{2} - 4n$ . Note that if  $\dim \mathcal{N}_{k-3}(\mathcal{P}) = N$ , then  $\ell_k = 0$ .

**Lemma 2.** *If  $k = 2n + 3$ , then  $\dim \mathcal{N}_{2n}(\mathcal{P}) = N$ .*

*Proof.* Assume  $\dim \mathcal{N}_{2n}(\mathcal{P}) \geq N + 1$ . We take  $4(n-r) - 3$  distinct points  $\mathcal{Q}_r$  on  $C_{n-r} \setminus \mathcal{P}_{n-r}$  for  $r = 0, \dots, n-1$ . Put  $\mathcal{Q} := \mathcal{Q}_0 \cup \dots \cup \mathcal{Q}_{n-1} \cup \{R\}$  where  $R \notin C_1$ . Then  $\sharp \mathcal{Q} = N$  and  $\dim \mathcal{N}_{2n}(\mathcal{P}, \mathcal{Q}) \geq \mathcal{N}_{2n}(\mathcal{P}) - N = 1$ . Hence we can take a non-zero element  $D \in \mathcal{N}_{2n}(\mathcal{P}, \mathcal{Q})$ . As  $\mathcal{Q}_0 \subset C_n \setminus \mathcal{P}_n$ , we have  $I(D, C_n) \geq 4 + 4n - 3 = 2 \cdot 2n + 1$ . Hence  $D \in C_n \mathcal{N}_{2n-2}(\mathcal{P}', \mathcal{Q}')$  where  $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_n$  and  $\mathcal{Q}' = \mathcal{Q} \setminus \mathcal{Q}_0$ . By the same argument for  $r = 1, \dots, n-2$ , then  $D$  is contained in  $C_n C_{n-1} \dots C_2 \mathcal{N}_2(\mathcal{P}_1, \mathcal{Q}_{n-1}, R)$ . But  $\mathcal{N}_2(\mathcal{P}_1, \mathcal{Q}_{n-1}, R) = \{0\}$  because  $R \notin C_1$ . This is a contradiction.  $\square$

**Lemma 3.** *If  $n \geq 3$  and  $k = 2n + 2$ , then  $\dim \mathcal{N}_{2n-1}(\mathcal{P}) = N$ .*

*Proof.* We assume  $\dim \mathcal{N}_{2n-1}(\mathcal{P}) \geq N + 1$ . We divide our considerations into two cases  $\dim \mathcal{N}_2(\mathcal{P}_{ij}) = 0$  for all  $(i, j)$  or  $\dim \mathcal{N}_2(\mathcal{P}_{ij}) \geq 1$  for some  $(i, j)$ . The first case is proved by the same argument of Lemma 2.

Now we consider the second case. We may assume that  $(i, j) = (1, 2)$  and we take a non-zero conic  $D_2 \in \mathcal{N}_2(\mathcal{P}_{12})$ . We take  $4n-9$  distinct points  $\mathcal{Q}_0$  on  $D_2 \setminus \mathcal{P}_{12}$  and  $4(n-r)-9$  distinct points  $\mathcal{Q}_r$  on  $C_{n-r+1} \setminus \mathcal{P}_{n-r+1}$  for  $r = 1, \dots, n-3$ . Put  $\mathcal{Q} = \mathcal{Q}_0 \cup \dots \cup \mathcal{Q}_{n-3} \cup \{R_1, R_2, R_3\}$  where  $R_1, R_2$  and  $R_3$  are not collinear. Then  $\sharp \mathcal{Q} = N$  and  $\dim \mathcal{N}_{2n-1}(\mathcal{P}, \mathcal{Q}) \geq \mathcal{N}_{2n-1}(\mathcal{P}) - N = 1$ . Hence we can take a non-zero element  $D \in \mathcal{N}_{2n-1}(\mathcal{P}, \mathcal{Q})$ . By the same argument,  $D$  is in  $D_2 C_n \dots C_3 \mathcal{N}_1(R_1, R_2, R_3)$ . But  $\mathcal{N}_1(R_1, R_2, R_3) = \{0\}$  because  $R_1, R_2$  and  $R_3$  are not collinear.  $\square$

**Corollary 1.**  $\widetilde{\text{Alex}}(C)_4$  is trivial.

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